Quasi-classical Limit of a Spin Coupled to a Reservoir

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Mathematical and Philosophical Foundations of Open Systems @ DMAT

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1. Introduction:
   - Spin-boson model [CFFM,FM] describing a spin interacting with a bosonic field in the quasi-classical regime [CF,CFO1-2].
2. Main result [CFFM]:
   - Derivation of the effective dynamics for the small system;
   - Analysis of decoherence and markovianity in the energy preserving case;
   - Energy exchange case.
3. Future perspectives [CFF].

Main References

- [CF] M.C., M. Falconi, *Ann. H. Poincaré* **19** (2018);
- [CFFM] M.C., M. Falconi, M. Fantechi, M. Merkli, *arXiv:2408.02515*;
- [CFO1] M.C., M. Falconi, M. Olivieri, *J. Eur. Math. Soc.* **25** (2023);
- [CFO2] M.C., M. Falconi, M. Olivieri, *Anal. PDE* **16** (2023);
- [CFF] M.C., M. Falconi, M. Fantechi, in preparation.
- [FM] M. Fantechi, M. Merkli, preprint *arXiv:2409.15850*;

Open Quantum System (I)

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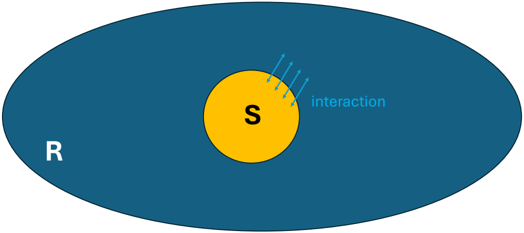

- A *small* quantum systems $\mathcal{S}$ interacts with a *large* (quasi-classical) reservoir or environment $\mathcal{R}$;
- We are interested in the effective behavior of $\mathcal{S}$ when the reservoir's degrees of freedom are traced out.

Open Quantum System (II)

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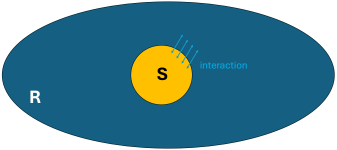

- Space of states $\mathscr{H} = \mathscr{H}_{\mathcal{S}} \otimes \mathscr{H}_{\mathcal{R}}$;
- Hamiltonian
  $H = H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathcal{R}} + H_I$.

Effective dynamics

Given an initial state $\rho \in \mathscr{L}^1_{+,1}(\mathscr{H})$, the reduced density matrix is

$$\gamma(t) = \mathrm{tr}_{\mathcal{R}} \left(e^{-itH} \rho e^{itH} \right), \qquad t \in \mathbb{R}.$$

Large-time behavior

The goal is to study the behavior of $\gamma(t)$ (and in particular its off-diagonal part) for large times and for different initial states ρ.

Open Quantum System (III)

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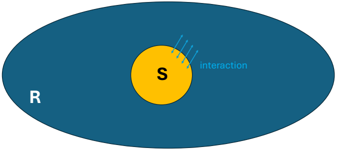

- Space of states $\mathscr{H} = \mathscr{H}_{\mathcal{S}} \otimes \mathscr{H}_{\mathcal{R}}$;
- $H = H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathcal{R}} + H_I$:
- $\gamma(t) = \mathrm{tr}_{\mathcal{R}} \left(e^{-itH} \rho e^{itH} \right)$.

Markovian approximation

- In the weak coupling approximation (a.k.a. Van Hove regime), $H_I = \lambda V$, for some small $|\lambda| \ll 1$;
- For times $t \lesssim \frac{1}{\lambda^2}$, the effective dynamics is given by a CPTP map with Linbladian generator [Davies '74];
- Under suitable assumptions on the model, the Markovian approximation can be proven to hold uniformly in time [Merkli '22].

Is it possible to go *beyond* the Markovian approximation?

Spin-Boson Model (I)

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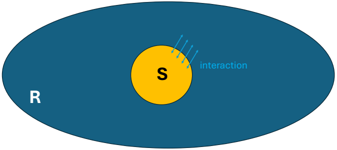

- Space of states $\mathscr{H} = \mathscr{H}_{\mathcal{S}} \otimes \mathscr{H}_{\mathcal{R}}$;
- $H = H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathcal{R}} + H_I$:
- $\gamma(t) = \mathrm{tr}_{\mathcal{R}} \left(e^{-itH} \rho e^{itH} \right)$.

Small system $\mathcal{S}$

- Two-level system $\mathscr{H}_{\mathcal{S}} = \mathbb{C}^2$;
- $H_{\mathcal{S}} = \frac{1}{2}\omega_0 \sigma_z$, with $\omega_0 > 0$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the 3rd Pauli matrix.

Reservoir $\mathcal{R}$

- Bosonic quantum field $\mathscr{H}_{\mathcal{R}} = \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^3)) = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^3)^{\otimes_{\mathrm{s}} n}$;
- $H_{\mathcal{R}} = \mathrm{d}\Gamma(\omega) = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, \omega(\mathbf{k}) a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, with dispersion relation $\omega(\mathbf{k}) \geqslant 0$.

Spin-Boson Model (II)

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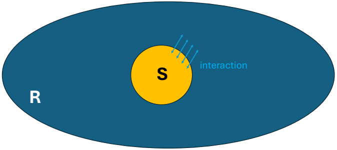

- Space of states $\mathscr{H} = \mathscr{H}_{\mathcal{S}} \otimes \mathscr{H}_{\mathcal{R}}$;
- $H = H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathcal{R}} + H_I$:
- $\gamma(t) = \mathrm{tr}_{\mathcal{R}}\left(e^{-itH} \rho e^{itH}\right)$.

Interaction

Linear coupling

$$H_I = \lambda \sqrt{\varepsilon}\, G \otimes \varphi(g).$$

- $\lambda \in \mathbb{R}$ is a coupling parameter;
- $0 < \varepsilon \leqslant 1$ is a quasi-classical parameter interpolating between quantum ($\varepsilon = 1$) and classical ($\varepsilon = 0$) reservoirs;
- $G \in \mathscr{B}(\mathbb{C}^2)$ is self-adjoint;
- field operator $\varphi(g) = \frac{1}{\sqrt{2}}\left[a^\dagger(g) + a(g)\right]$ with form factor $g \in L^2(\mathbb{R}^3)$.

Quasi-classical Scaling (I)

$$\mathscr{H} = \mathbb{C}^2 \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^3)), \qquad H = \tfrac{1}{2}\omega_0 \sigma_z + \mathrm{d}\Gamma(\omega) + \lambda\sqrt{\varepsilon} G \otimes \varphi(g).$$

Quasi-classical initial states

We want to consider initial states $\rho \in \mathscr{L}^1_{+,1}(\mathscr{H})$ of the form

$$\rho = \gamma \otimes \zeta_\varepsilon,$$

where the field's state contains a macroscopic average number of excitations $\bar{N} \sim \frac{1}{\varepsilon}$ as $\varepsilon \to 0$: with $N = \mathrm{d}\Gamma(1)$,

$$\mathrm{tr}_{\mathcal{R}}\left(N \zeta_\varepsilon\right) \sim \frac{1}{\varepsilon}.$$

This is a semiclassical regime for the field since

$$\left[a(g), a^\dagger(g)\right] = 1 \ll \bar{N} \sim \frac{1}{\varepsilon}.$$

Quasi-classical Scaling (II)

$$\mathscr{H} = \mathbb{C}^2 \otimes \Gamma_s(L^2(\mathbb{R}^3)), \qquad H = \tfrac{1}{2}\omega_0\sigma_z + \mathrm{d}\Gamma(\omega) + \lambda\sqrt{\varepsilon}G \otimes \varphi(g).$$

Quasi-classical variables

To make the emergence of the classical behavior more apparent, it is convenient to rescale all field's observable

$$a^\sharp(g) \longrightarrow a^\sharp_\varepsilon(g) := a^\sharp(\sqrt{\varepsilon}g) = \sqrt{\varepsilon}a^\sharp(g),$$

(and consequently $\varphi_\varepsilon(g) := \varphi(\sqrt{\varepsilon}g)$), so that

$$\lim_{\varepsilon\to 0} \mathrm{tr}_{\mathcal{R}}\left(A_\varepsilon B_\varepsilon\right) = \lim_{\varepsilon\to 0} \mathrm{tr}_{\mathcal{R}}\left(B_\varepsilon A_\varepsilon\right) \quad \Longleftarrow \quad [A_\varepsilon, B_\varepsilon] = \mathcal{O}(\varepsilon),$$

for any polynomials $A_\varepsilon, B_\varepsilon$ in $a^\sharp_\varepsilon$.
The Hamiltonian becomes $H = H_0 + \lambda G \otimes \varphi_\varepsilon(g)$ where the free part reads

$$H_0 = \tfrac{1}{2}\omega_0\sigma_z + \tfrac{1}{\varepsilon}\mathrm{d}\Gamma_\varepsilon(\omega)$$

Initial States (I)

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$$\mathscr{H} = \mathbb{C}^2 \otimes \Gamma_s(L^2(\mathbb{R}^3)), \qquad \rho(0) = \gamma \otimes \zeta_\varepsilon.$$

Quasi-classical states

- The classical counterpart of field's states ζ_ε are probability measures $\mathscr{M}(L^2(\mathbb{R}^3))$ over the one-excitation space $L^2(\mathbb{R}^3)$;
- The generating functional $\chi_\varepsilon : L^2(\mathbb{R}^3) \to \mathbb{C}$ associated to ζ_ε is

$$\chi_\varepsilon(f) := \mathrm{tr}_{\mathcal{R}} \left(W_\varepsilon(f) \zeta_\varepsilon \right), \qquad W_\varepsilon(f) = e^{i\varphi_\varepsilon(f)}, f \in L^2(\mathbb{R}^3).$$

Quasi-classical convergence

We say that $\zeta_\varepsilon \xrightarrow[\varepsilon \to 0]{\mathrm{qc}} \mu \in \mathscr{M}_{\mathrm{cyl}}(L^2(\mathbb{R}^3))$ if

$$\chi_\varepsilon(f) \xrightarrow[\varepsilon \to 0]{\mathrm{qc}} \int_{L^2(\mathbb{R}^3)} \mathrm{d}\mu(g)\, e^{i\sqrt{2}\Re\langle g | f \rangle}.$$

If, for some $\delta > 0$, $\mathrm{tr}_{\mathcal{R}} \left(N_\varepsilon^\delta \zeta_\varepsilon \right) \leqslant C$, then $\mu \in \mathscr{M}(L^2(\mathbb{R}^3))$.

Initial States (II)

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$$\rho(0) = \gamma \otimes \zeta_\varepsilon, \qquad \chi_\varepsilon(f) := \mathrm{tr}_{\mathcal{R}}\left(e^{i\varphi_\varepsilon(f)}\zeta_\varepsilon\right).$$

(Superposition of) coherent states

- Coherent states of the form

$$\Psi_{f_0} := e^{\frac{\sqrt{2}\varphi_\varepsilon(f_0)}{\sqrt{\varepsilon}}}\Omega = W_\varepsilon\left(-\frac{\sqrt{2}if_0}{\sqrt{\varepsilon}}\right)\Omega,$$

  have a simple counterpart: $\chi_\varepsilon(f) = e^{-\frac{1}{4}\varepsilon\|f\|^2} e^{i\sqrt{2}\Re\langle f_0|f\rangle}$ so that

$$|\Psi_{f_0}\rangle\langle\Psi_{f_0}| \xrightarrow[\varepsilon\to 0]{\mathrm{qc}} \delta(\cdot - f_0).$$

- More in general for a superposition of coherent states

$$\zeta_\varepsilon = \int_{L^2(\mathbb{R}^3)} \mathrm{d}\mu_0(f)\,|\Psi_f\rangle\langle\Psi_f|,$$

  for some $\mu_0 \in \mathscr{M}(L^2(\mathbb{R}^3))$, then

$$\zeta_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathrm{qc}} \mu_0.$$

Initial States (III)

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$$\rho(0) = \gamma \otimes \zeta_\varepsilon, \qquad \chi_\varepsilon(f) := \mathrm{tr}_{\mathcal{R}} \left(e^{i\varphi_\varepsilon(f)} \zeta_\varepsilon \right).$$

Bose-Einstein condensate

- For a given one-excitation state $f_0 \in L^2(\mathbb{R}^3)$, a Bose-Einstein condensate is described by the product state $f_0 \otimes \cdots \otimes f_0$ or

$$\psi_\varepsilon = \frac{a^\dagger(f_0)^n}{\sqrt{n!}} \Omega, \qquad n = \lfloor 1/\varepsilon \rfloor.$$

- The classical measure is the uniform over the sphere S^1:

$$\zeta_\varepsilon \xrightarrow[\varepsilon \to 0]{\mathrm{qc}} \int_0^{2\pi} \mathrm{d}\theta \, \delta \left(\cdot - f_0 e^{-i\theta} \right).$$

Proof.

$\chi_\varepsilon(f) = L_n \left(\tfrac{1}{2} \varepsilon |\langle f_0 | f \rangle|^2 \right) \langle \Omega | W_\varepsilon(f) \Omega \rangle \to J_0 \left(\sqrt{2} |\langle f_0 | f \rangle| \right)$, but the integral form of Bessel functions yields $\displaystyle\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \, e^{i\sqrt{2}\,\Re\left(e^{i\theta}\langle f_0|f\rangle\right)}$. □

Initial States (IV)

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$$\rho(0) = \gamma \otimes \zeta_\varepsilon, \qquad \chi_\varepsilon(f) := \mathrm{tr}_{\mathcal{R}} \left(e^{i\varphi_\varepsilon(f)} \zeta_\varepsilon \right).$$

Thermal state

- The thermal equilibrium state of $\mathcal{R}$ is identified by
$$\left\langle a^\dagger(f) a(g) \right\rangle_\beta = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \, \frac{1}{e^{\beta\omega(k)} - 1} g^*(\mathbf{k}) f(\mathbf{k}).$$
- To obtain a non-trivial limit we are forced to rescale the temperature and set $\beta = \varepsilon\beta'$, for β' independent of ε (high temperature regime).
- $\zeta_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathrm{qc}} \mu \in \mathscr{M}_{\mathrm{cyl}}(L^2(\mathbb{R}^3))$ gaussian measure with zero mean and covariance $\beta'\omega$ (in ∞ dimension the gaussian measure is cylindrical).

Proof.

$$\chi_\varepsilon(f) = e^{-\frac{1}{4}\varepsilon\langle f | \coth(\varepsilon\beta'\omega/2) f \rangle} \xrightarrow[\varepsilon\to 0]{} e^{-\frac{1}{2\beta'}\left\langle f \middle| \omega^{-1} f \right\rangle}.$$

□

Main Goals

$$\rho(0) = \gamma \otimes \zeta_\varepsilon, \qquad H = \tfrac{1}{2}\omega_0 \sigma_z + \tfrac{1}{\varepsilon}\mathrm{d}\Gamma_\varepsilon(\omega) + \lambda\sqrt{\varepsilon} G \otimes \varphi(g).$$

Main Goals

Characterize the (large-time) effective dynamics of the spin $\mathcal{S}$, i.e., the behavior for $t \to +\infty$ of

$$\gamma_\varepsilon(t) = \mathrm{tr}_{\mathcal{R}}\left[e^{-itH(\varepsilon)}(\gamma \otimes \zeta_\varepsilon)e^{itH(\varepsilon)}\right],$$

for $\varepsilon \in (0, 1]$ and as $\varepsilon \to 0$, and for different field's initial states ζ_ε – COHerent superposition, Bose-Einstein Condensate, THerMal state. In particular, we aim to

- investigate decoherence, i.e., suppression of interference effects;
- study the markovianity of the effective dynamics, i.e., how much distinguishability between states gets worse along the dynamics.

Effective Dynamics

$$\rho(0) = \gamma \otimes \zeta_\varepsilon, \qquad H = \tfrac{1}{2}\omega_0 \sigma_z + \tfrac{1}{\varepsilon} \mathrm{d}\Gamma_\varepsilon(\omega) + \lambda G \otimes \varphi_\varepsilon(g).$$

Theorem (MC, Falconi, Olivieri '23)

Let $\zeta_\varepsilon \in \mathscr{L}^1_{+,1}(L^2(\mathbb{R}^2))$ *be a family of field's states such that, for some* $\delta > 0$, $\mathrm{tr}_{\mathcal{R}}\left(N_\varepsilon^\delta \zeta_\varepsilon\right) \leqslant C$, *then* $\exists$ *a probability measure* $\mu_0 \in \mathscr{M}(L^2(\mathbb{R}^2))$ *and a subsequence* $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\varepsilon_k \xrightarrow[k\to+\infty]{} 0$, *satisfying* $\zeta_{\varepsilon_k} \xrightarrow[k\to+\infty]{\mathrm{qc}} \mu_0$ *and*

$$\lim_{k\to+\infty} \gamma_{\varepsilon_k}(t) = \gamma_0(t) := \int_{L^2(\mathbb{R}^3)} \mathrm{d}\mu_0(f)\, U_t(f) \gamma U_t^\dagger(f),$$

where $U_t(f) := U_{t,0}(f)$ *is the two-parameter unitary group*

$$i\partial_t U_{t,s}(f) = \left(H_{\mathcal{S}} + \sqrt{2}\lambda G\, \Re \left\langle e^{-it\omega} f \middle| g \right\rangle\right) U_{t,s}(f), \qquad U_{t,t}(f) = \mathbb{1}.$$

Energy Preserving Model

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$$\rho(0) = \gamma \otimes \zeta_\varepsilon, \qquad H = \tfrac{1}{2}\omega_0\sigma_z + \tfrac{1}{\varepsilon}\mathrm{d}\Gamma_\varepsilon(\omega) + \tfrac{1}{2}\lambda\sigma_z \otimes \varphi_\varepsilon(g).$$

Effective dynamics

- The effective dynamics is explicitly solvable $\forall \varepsilon \in (0,1]$ in the basis $|1\rangle, |2\rangle$ s.t. $\sigma_z|1\rangle = |1\rangle$, $\sigma_z|2\rangle = -|2\rangle$: we set $[\gamma(t)]_{ij} := \langle i|\gamma(t)|j\rangle$;
- The diagonal terms are constant: $[\gamma(t)]_{ii} = \gamma_{ii}$, $\forall i$ and $\forall t \in \mathcal{R}$.

Proposition (Effective dynamics)

For all $\varepsilon > 0$ and for all $t \geq 0$,

$$[\gamma_\varepsilon(t)]_{12} = e^{-i\omega_0 t} D_\varepsilon(t)\gamma_{12},$$

where the decoherence function is given by (with generating functional χ_ε)

$$D_\varepsilon(t) = \chi_\varepsilon(\lambda g_t), \qquad g_t(\mathbf{k}) := \frac{1 - e^{i\omega(\mathbf{k})t}}{i\omega(\mathbf{k})} g(\mathbf{k}).$$

EPM Decoherence (I)

$$[\gamma_\varepsilon(t)]_{12} = e^{-i\omega_0 t} D_\varepsilon(t)\gamma_{12}, \qquad D_\varepsilon(t) = \chi_\varepsilon\left(\lambda g_t\right), \quad g_t(\mathbf{k}) := \frac{1 - e^{i\omega(\mathbf{k})t}}{i\omega(\mathbf{k})} g(\mathbf{k}).$$

Decoherence function

The decoherence function can be explicitly computed:

- (COH)

$$D_\varepsilon(t) = e^{-\frac{1}{4}\varepsilon\lambda^2 \|g_t\|^2} \int_{L^2(\mathbb{R}^3)} \mathrm{d}\mu_0(f)\, e^{i\sqrt{2}\lambda \Re\langle f | g_t\rangle};$$

- (BEC) with $n = \left\lfloor \frac{1}{\varepsilon} \right\rfloor$,

$$D_\varepsilon(t) = e^{-\frac{1}{4}\varepsilon\lambda^2 \|g_t\|^2}\, L_n\left(\tfrac{1}{2}\,\varepsilon\lambda^2\, |\langle f_0 | g_t\rangle|^2\right),$$

$$D_0(t) = J_0\left(\sqrt{2}|\lambda|\,|\langle f_0 | g_t\rangle|\right) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi}\, e^{i\sqrt{2}\lambda\Re\left(e^{i\theta}\langle f | g_t\rangle\right)};$$

- (THM)

$$D_\varepsilon(t) = e^{-\frac{1}{4}\varepsilon\lambda^2 \langle g_t | \coth(\beta'\varepsilon\omega/2) g_t\rangle}.$$

EPM Decoherence (II)

$$[\gamma_\varepsilon(t)]_{12} = e^{-i\omega_0 t} D_\varepsilon(t)\gamma_{12}, \qquad D_\varepsilon(t) = \chi_\varepsilon(\lambda g_t), \quad g_t(\mathbf{k}) := \frac{1 - e^{i\omega(\mathbf{k})t}}{i\omega(\mathbf{k})} g(\mathbf{k}).$$

Proposition (Decoherence)

Let $\mathcal{J}(k) = \frac{\pi}{2} k^2 \int_{S^2} \mathrm{d}\Sigma(\vartheta,\phi)\ |g(k,\vartheta,\phi)|^2$ *be the spectral density of* $\mathcal{R}$*, and assume that* $\mathcal{J}(k) \sim k^p$*, as* $k \to 0^+$*, for some* $p \in (-1, +\infty)$*. Then,*

1. *(COH, BEC) for quantum* $\mathcal{R}$ *(*$\varepsilon > 0$*), there is full decoherence:* $\lim_{t\to\infty} D_\varepsilon(t) = 0$*;*
2. *(COH, BEC) for classical* $\mathcal{R}$ *(*$\varepsilon = 0$*), there is partial decoherence only:* $\lim_{t\to\infty} D_0(t) = D_0(\infty) \neq 0$*;*
3. *(THM) For both quantum or classical* $\mathcal{R}$*, there is partial decoherence if* $p > 2$*, and full decoherence if* $0 < p < 2$*; furthermore,*

$$\frac{D_\varepsilon(t)}{D_0(t)} \propto e^{-\frac{1}{\pi}\Gamma_\infty \varepsilon \lambda}.$$

EPM Decoherence (III)

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$$[\gamma_\varepsilon(t)]_{12} = e^{-i\omega_0 t} D_\varepsilon(t)\gamma_{12}, \qquad D_\varepsilon(t) = \chi_\varepsilon(\lambda g_t), \quad g_t(\mathbf{k}) := \frac{1 - e^{i\omega(\mathbf{k})t}}{i\omega(\mathbf{k})} g(\mathbf{k}).$$

(COH) Decoherence

For a single coherent state Ψ_{f_0}, $D_0(t) = 1$, $\forall t \in \mathbb{R}$ (no decoherence), while $D_\varepsilon(t) \xrightarrow[t\to+\infty]{} 0$, $\forall \varepsilon > 0$ (full decoherence).

(BEC) Decoherence

Since $D_0(t) = J_0\left(\sqrt{2}|\lambda|\,|\langle f_0|g_t\rangle|\right) < 1$, there is always decoherence but typically no full decoherence: if $|\left\langle f_0|\omega^{-1}g\right\rangle| < +\infty$, then

$$\lim_{t\to\infty} D_0(t) = J_0\left(\sqrt{2}|\lambda|\left|\left\langle f_0\left|\tfrac{1}{\omega}g\right.\right\rangle\right|\right),$$

which vanishes only if the argument is a zero of J_0. Hence, typically there is full decoherence for quantum $\mathcal{R}$ and partial decoherence for classical $\mathcal{R}$.

EPM Markovianity

Measure of markovianity

In [Laino, Piilo, Breuer '10] a quantitative measure of markovianity has been introduced:

- The key idea is that the distinguishability of two quantum states ρ, ν never decreases along a markovian dynamics;
- A measure of how much two states differ is $\|\rho - \nu\|_1$;
- One thus defines the measure of markovianity of the effective dynamics as

$$\mathcal{N} = \max_{\rho(0),\nu(0)} \int_{S_+} \mathrm{d}t \, \partial_t \left\| \rho(t) - \nu(t) \right\|_1 ,$$

  where $S_+ = \{t \geq 0, \partial_t \left\| \rho(t) - \nu(t) \right\|_1 > 0\}$.

- In our case $S_+ = \{t \geq 0, \partial_t |D_\varepsilon(t)| > 0\}$ and $\partial_t |D_\varepsilon(t)|$ can be explicitly computed...

EPM Markovianity – BEC (I)

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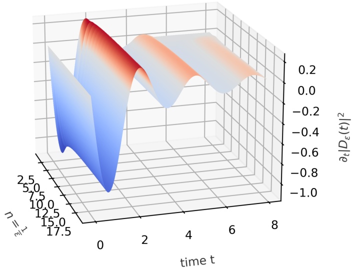

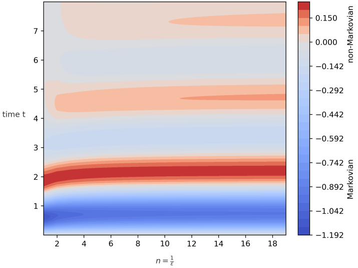

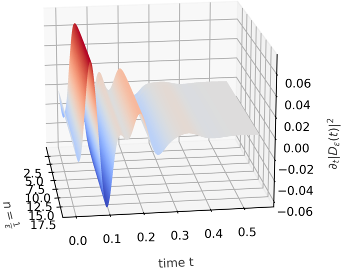

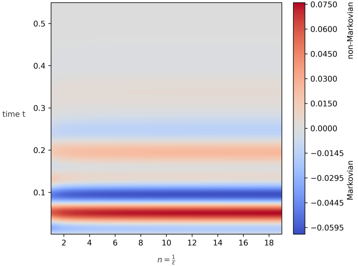

EPM Markovianity – BEC (II)

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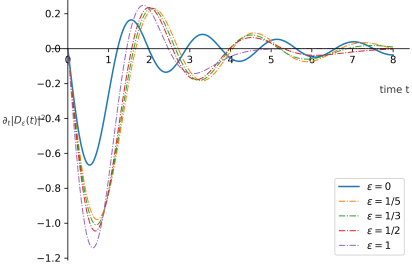
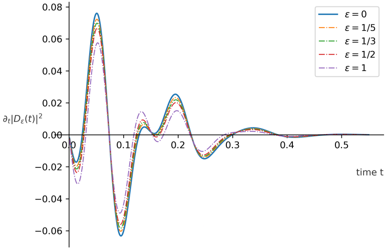
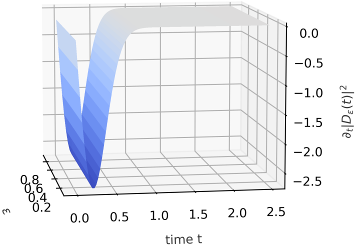


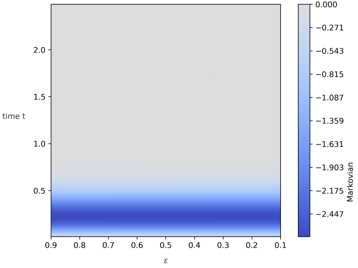


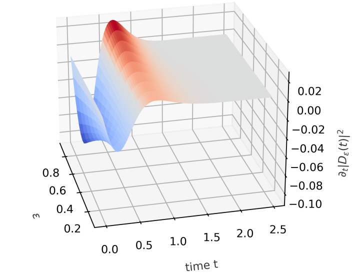
Markovianity

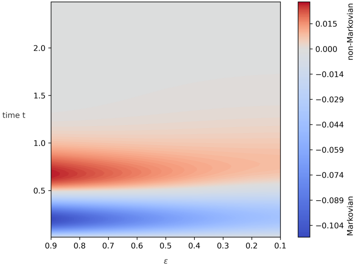
- There are oscillations with decreasing amplitude and frequency depending on the infrared behavior of $\mathcal{R}$;
- As ε decreases from $\varepsilon = 1$ (quantum) to $\varepsilon = 0$ (classical), the regions of non-Markovianity stabilize to become ε-independent.

EPM Markovianity – THM









Energy Exchange Model

$$\rho(0) = \gamma \otimes \zeta_\varepsilon, \qquad H = \tfrac{1}{2}\omega_0\sigma_z + \tfrac{1}{\varepsilon}\mathrm{d}\Gamma_\varepsilon(\omega) + \lambda\sqrt{\varepsilon}G \otimes \varphi(g).$$

Effective dynamics

- The quantum model is no longer diagonalizable nor explicitly solvable;
- The effective quasi-classical dynamics is given by

$$\gamma_0(t) = \int_{L^2(\mathbb{R}^3)} \mathrm{d}\mu_0(f)\, U_t(f)\gamma U_t^\dagger(f),$$

where $U_t(f) := U_{t,0}(f)$ is the two-parameter unitary group

$$i\partial_t U_{t,s}(f) = \left(H_\mathcal{S} + \sqrt{2}\lambda G\, \alpha_t(f)\right) U_{t,s}(f), \qquad U_{t,t}(f) = \mathbb{1},$$
$$\alpha_t(f) = \Re \left\langle e^{-it\omega} f \middle| g \right\rangle.$$

- Qualitatively, if $\alpha_t(f) \xrightarrow[t\to\infty]{} 0$ (Riemann-Lebesgue lemma), then the interaction becomes small asymptotically (scattering).

EEM Scattering (I)

$$i\partial_t U_{t,s}(f) = \left(H_{\mathcal{S}} + \sqrt{2}\lambda G\, \alpha_t(f)\right) U_{t,s}(f), \qquad \alpha_t(f) = \Re \left\langle e^{-it\omega} f \middle| g \right\rangle.$$

Scattering

- Let $\Omega_+ = \lim_{t \to +\infty} U_{t,0}^{\dagger} e^{-itH_{\mathcal{S}}}$ and Ω_- be the wave operators;
- Set $a(f) := \int_0^{+\infty} \mathrm{d}t \; |\alpha_t(f)|$.
- If the conditions $a(f) < +\infty$, $|\lambda|\, a(f) < 1$, are satisfied by any $f \in \mathrm{supp}(\mu_0)$, the $\mathcal{S}$ effective dynamics is asymptotically free.

Proposition (Existence of scattering)

Let $f \in L^2(\mathbb{R}^3)$ be fixed and suppose that $a(f) < +\infty$. Then the wave operators $\Omega_\pm$ exist. Moreover, if $|\lambda|\, a(f) < 1$, then $\Omega_\pm$ are invertible and the scattering operator $S = \Omega_+^{-1}\Omega_-$ exists.

EEM Scattering (II)

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$$\gamma_0(t) = \int_{L^2(\mathbb{R}^3)} \mathrm{d}\mu_0(f)\, U_t(f)\gamma U_t^\dagger(f), \qquad a(f) := \int_0^{+\infty} \mathrm{d}t \left| \Re \left\langle e^{-it\omega} f \middle| g \right\rangle \right|.$$

Proposition (Stability of the free dynamics)

Let $\zeta_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathrm{qc}} \mu_0$ *and* $a(f) < +\infty$, $\forall f \in \mathrm{supp}(\mu_0)$. *Then,* $\exists C < +\infty$ *s.t.*

$$\sup_{t\geq 0} \left\| \gamma_0(t) - e^{-itH_S}\gamma e^{itH_S} \right\| \leq C|\lambda|.$$

Lemma

Let $\omega(k) = |\mathbf{k}|$, *with polar coordinates* $(k, \Sigma) \in \mathbb{R}^+ \times S^2$. *If*

- $k \mapsto f(k,\Sigma)g(k,\Sigma)$ *is* $C^2(\mathbb{R}^+)$ *for any* Σ,
- *for any* Σ, $\begin{cases} f(k,\Sigma)g(k,\Sigma) \sim k^p, & \text{as } k \to 0, \text{ for some } p > -1, \\ f(k,\Sigma)g(k,\Sigma) \sim k^{-q} & \text{as } k \to +\infty, \text{ for some } q > 2, \end{cases}$

then $a(f) < +\infty$.

EEM Non-scattering

$$i\partial_t U_{t,s}(f) = \left(H_{\mathcal{S}} + \sqrt{2}\lambda G\, \alpha_t(f)\right) U_{t,s}(f), \qquad \alpha_t(f) = \Re \left\langle e^{-it\omega} f \middle| g \right\rangle.$$

Polaron-type models

- For polarons $\omega(\mathbf{k}) = \omega_{\mathcal{R}}$ and $\alpha_t(f) = \Re\, e^{i\omega_{\mathcal{R}} t} \langle f|g\rangle$ is time-periodic;
- The effective dynamics is given by a well-known equation describing open quantum systems interacting with classical fields:

$$i\partial_t U_t(f) = \left[\tfrac{1}{2}\omega_0\sigma_z + \tfrac{\lambda}{\sqrt{2}} \langle f|g\rangle \cos(\omega_{\mathcal{R}} t)\sigma_x\right] U_t(f), \qquad U_0(f) = \mathbb{1}.$$

Proposition (Evolution in presence of symmetries)

Suppose that $\mu_0(A) = \mu_0(-A)$, $\forall A$ measurable and that G is off-diagonal in the σ_z-basis. Then the diagonal and the off-diagonal density matrix elements of $\gamma(t)$ evolve independently. In particular, if γ is diagonal, then so is $\gamma(t)$ for all t, and if γ is off-diagonal, then so is $\gamma(t)$.

Future Perspectives

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- Investigate further the energy exchange model (e.g. for a circularly polarized fields) in the non-scattering regime at least numerically;
- Study more realistic models, e.g., the Caldera-Leggett model, where a quantum trapped particle is linearly coupled to a bosonic field [MC, Falconi, Fantechi '2x].

Thank you for the attention!

EPM Decoherence (IV)

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$$[\gamma_\varepsilon(t)]_{12} = e^{-i\omega_0 t} D_\varepsilon(t)\gamma_{12}, \qquad D_\varepsilon(t) = \chi_\varepsilon\left(\lambda g_t\right), \quad g_t(\mathbf{k}) := \frac{1 - e^{i\omega(\mathbf{k})t}}{i\omega(\mathbf{k})} g(\mathbf{k}).$$

Sketch of the proof

- The key term for decoherence is the exponential factor $e^{-\frac{1}{4}\varepsilon\lambda^2 \|g_t\|^2}$;
- With the spectral density $\mathcal{J}(k) = \frac{\pi}{2} k^2 \int_{S^2} \mathrm{d}\Sigma(\vartheta, \phi)\ |g(k, \vartheta, \phi)|^2$, one has, e.g., for $\omega(\mathbf{k}) = |\mathbf{k}|^2$,

$$\|g_t\|^2 = \frac{4}{\pi} \int_0^{+\infty} \mathrm{d}k\, \frac{1 - \cos(kt)}{k^2} \mathcal{J}(k).$$

Large-time asymptotics

The $t \to \infty$ asymptotics of $\|g_t\|^2$ is determined by the infrared behavior of $\mathcal{J}$ as $k \to 0^+$. If $\mathcal{J}(k) = k^p \mathcal{I}(k)$, for some smooth non-vanishing $\mathcal{I}$, then one can classify all possible cases for $p \in (-1, +\infty)$ [Trushechkin '23].

EPM Decoherence (V)

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$$[\gamma_\varepsilon(t)]_{12} = e^{-i\omega_0 t} D_\varepsilon(t)\gamma_{12}, \qquad D_\varepsilon(t) = \chi_\varepsilon\left(\lambda g_t\right), \quad g_t(\mathbf{k}) := \frac{1 - e^{i\omega(\mathbf{k})t}}{i\omega(\mathbf{k})} g(\mathbf{k}).$$

Lemma

Let $\mathcal{J}(k) = k^p \mathcal{I}(k)$*, for some* $p \in (-1, +\infty)$ *and some* $\mathcal{I} \in C^2([0, k_c])$ *s.t.* $\mathcal{I}(0), \mathcal{I}'(0) \neq 0$*. Then, as* $t \to \infty$*,*

$$e^{-\frac{1}{4}\varepsilon\lambda^2 \|g_t\|^2} \sim \begin{cases} e^{-\frac{1}{\pi}\varepsilon\lambda\Gamma_\infty}, & p > 1, \\ c_0 t^{-c_1\varepsilon\lambda^2}, & p = 1, \\ c_0 t^{-c_2\varepsilon\lambda^2} e^{-c_1\varepsilon\lambda^2 t}, & p = 0, \\ c_0 e^{-c_1\varepsilon\lambda^2 t^{1-p}}, & -1 < p < 1, p \neq 0, \end{cases}$$

for some $\Gamma_\infty, c_1 > 0$ *and* $c_2 \in \mathcal{R}$ *(independent of* ε, λ*). Furthermore,*

$$c_0 = c_0(\varepsilon\lambda^2) > 0, \qquad \lim_{\varepsilon^2\lambda \to 0} c_0(\varepsilon^2\lambda) = 1.$$