Analogies and Disanalogies Between Classical and Quantum Physical Computation

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November 25, 2024

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I. Introduction

What is quantum computing?

- Combines fundamental concepts from two separate sciences: (1) Computer science and (2) Physics.

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- Quantum computing describes computational models which are specified using the abstract language associated with quantum mechanics (QM), as opposed to the abstract language associated with classical mechanics (CM).

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Noisy Intermediate-Scale Quantum computing (NISQ)

- NISQ is a near-term workaround that focuses on hardware components and problems that are inherently less sensitive to noise (e.g., simulation, sampling).

II. Computational complexity theory

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- Informal characterisation of human computation

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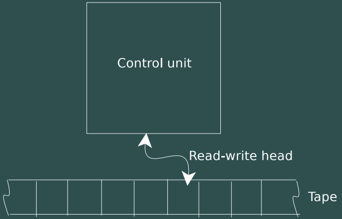

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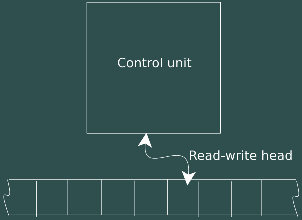

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- Turing machine: Captures what is essential about the process of human computation

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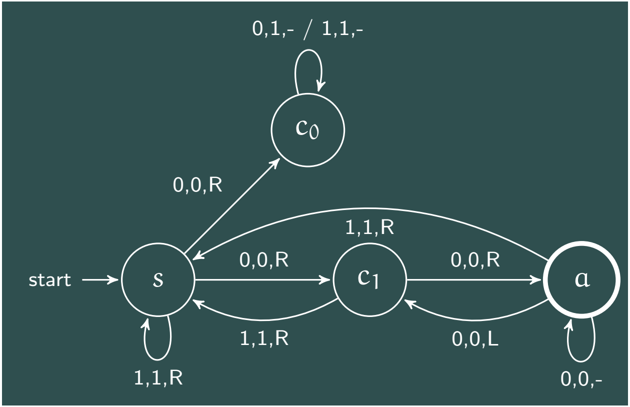
Example of a Nondeterministic Turing Machine (NTM)



Accepts: $*00$.

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$$\mathrm{NP} =_{df} \bigcup_{k \geq 1} \mathrm{NTIME}(n^k).$$

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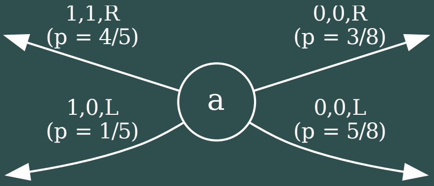

Probabilistic Turing machine (PTM)

- Probabilities are associated with transitions.

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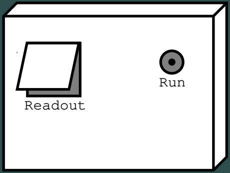
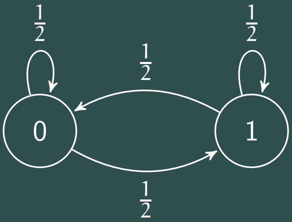
BQP (quantum bounded-error probabilistic polynomial time)

- Class of languages such that there exists a Quantum-probabilistic Turing machine that will produce an answer to the question of whether or not a given string x is in L in polynomial time, with the probability that it is correct, $p \geq 2/3$.


Readout
Run


1/2
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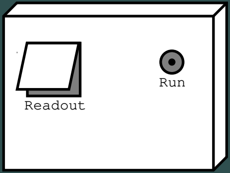
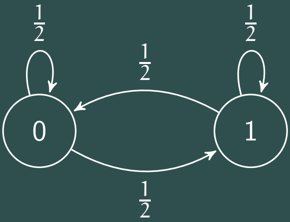
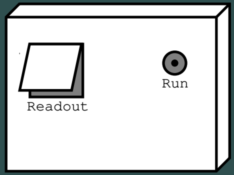
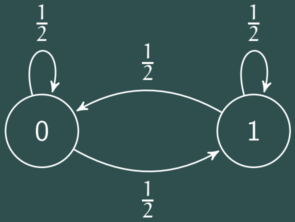




$\Pr(C0 \to 0)$





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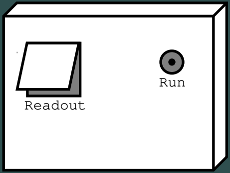

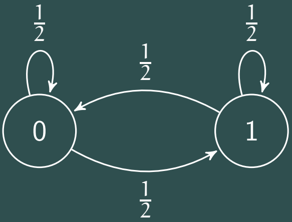

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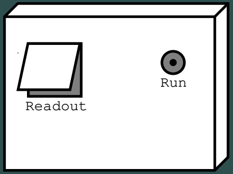

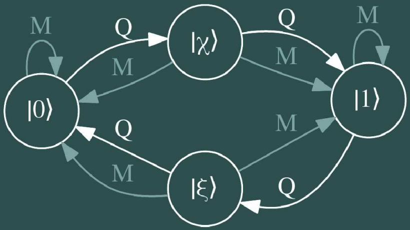

$$|0\rangle \xrightarrow{Q} \underbrace{\frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}_{|\chi\rangle} \xrightarrow{Q} |1\rangle$$

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$$\mathrm{BPP} \subsetneq \mathrm{BQP}?$$

$\Rightarrow$ Do problems exist that are tractable for a quantum computer but (provably) intractable for a PTM?

III. The Physics of Classical and Quantum Computers

Classical computers

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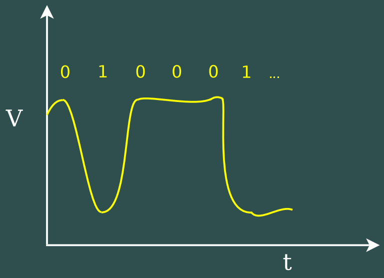
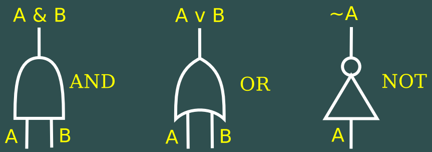


Classical logic gates



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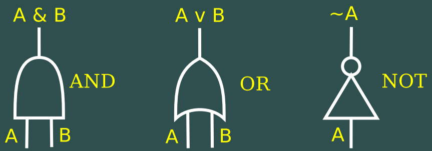
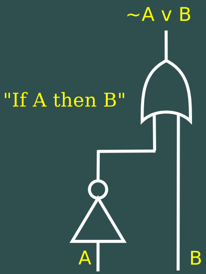
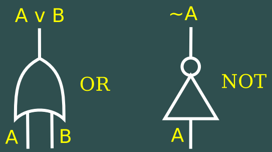


Classical circuit



Universal set:



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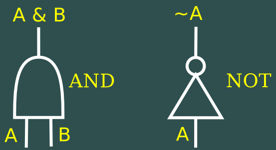

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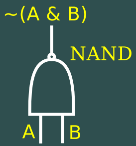

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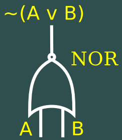

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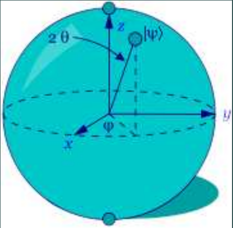

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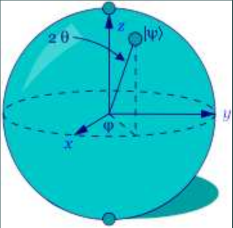

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- Typically initialized in one of the "computational basis" states, i.e., the eigenstates, $\{|0\rangle, |1\rangle\}$, of the Pauli Z operator.

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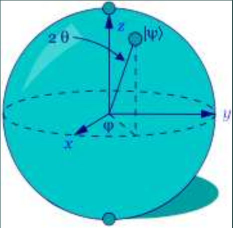

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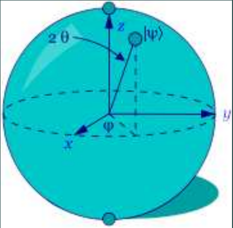
- Typically initialized in one of the "computational basis" states, i.e., the eigenstates, $\{|0\rangle, |1\rangle\}$, of the Pauli Z operator.
- General form for a qubit's state (in the z-basis): $\alpha|0\rangle + \beta|1\rangle$

Quantum computers

- Manipulate quantum bits (qubits), abstractly: unit vectors in a 2-dimensional complex Hilbert space.
- Physically realised by a 2-level quantum system, e.g., the spin of an electron in a particular direction.

$|0\rangle$



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- General form for a qubit's state (in any basis): $c_1|b_1\rangle + c_2|b_2\rangle$

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- Described by unitary transformations (reversible, norm preserving), $UU^{\dagger} = U^{\dagger}U = I$, on unit vectors

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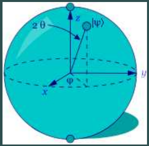
$$|0\rangle \;—\boxed{X}—\; |1\rangle$$

$$|1\rangle \;—\boxed{X}—\; |0\rangle$$

π-rotation about x-axis



$$X|0\rangle \;=\; |1\rangle \qquad\qquad X|1\rangle \;=\; |0\rangle$$

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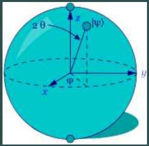
$$|0\rangle \;—\!\!\!—\boxed{Y}—\!\!\!—\; i|1\rangle$$

$$|1\rangle \;—\!\!\!—\boxed{Y}—\!\!\!—\; -i|0\rangle$$

π-rotation about y-axis



$$Y|0\rangle \;=\; i|1\rangle \qquad\qquad Y|1\rangle \;=\; -i|0\rangle$$

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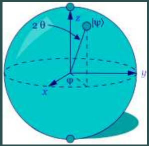
$$|0\rangle \;—\boxed{Z}—\; |0\rangle$$

$$|1\rangle \;—\boxed{Z}—\; -|1\rangle$$

π-rotation about z-axis



$$Z|0\rangle \;=\; |0\rangle \qquad\qquad Z|1\rangle \;=\; -|1\rangle$$

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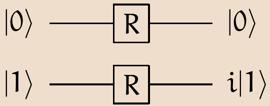
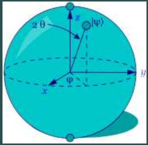


$\pi/2$-rotation about z-axis



$$R|0\rangle = |0\rangle \qquad R|1\rangle = i|1\rangle$$

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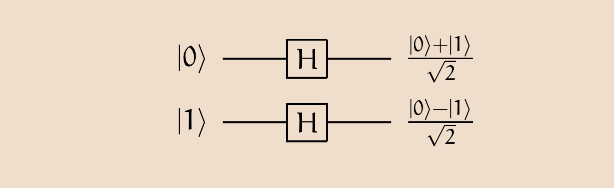
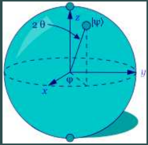


(1) $\pi/2$-rotation about y-axis +
(2) π-rotation about x



$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

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$$C_{10}|x\rangle|y\rangle = |x\rangle|y \oplus x\rangle$$

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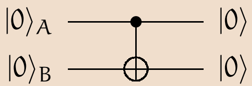

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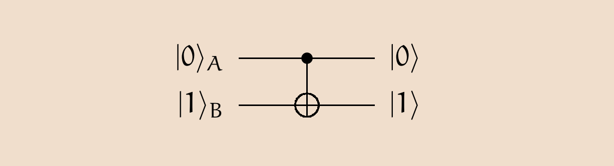

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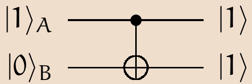

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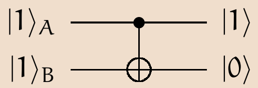
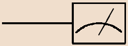


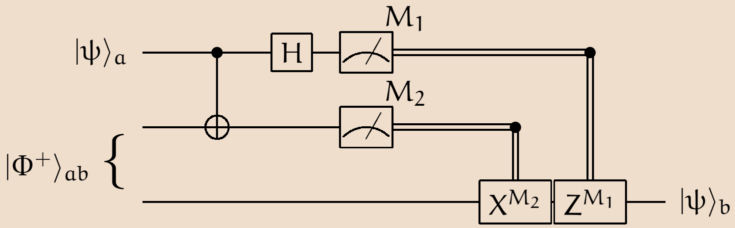
Measurement gate

Implemented by a projective measurement in a given basis.

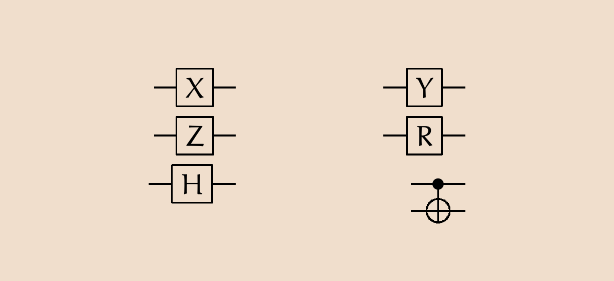
- Probabilistic transition
- Given $\alpha|0\rangle + \beta|1\rangle$, measurement in the z-basis yields $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$.

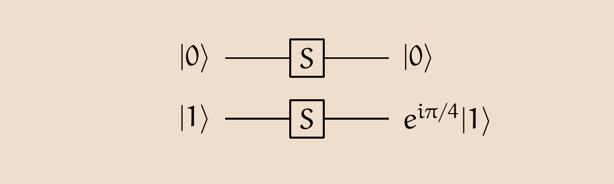


Circuits



Universal set of quantum logic gates:





Can simulate any quantum gate to arbitrary accuracy

IV. Deeper Analogies and Disanalogies

Consider the operator:

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- An observable A is represented by $f_A(\omega)$ acting on the phase space of a system.

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- The case in which a state description is always "complete" (in the classical sense) is a special case of the more general framework described by QM.
- Result: exponentially more possible state descriptions for a given n-dimensional system in QM. I.e., QM allows for entangled states.
- From a computational point of view: Exponentially more resources available – allows for "shortcuts" through a system's state space.

Illustration: General constraint on the correlations between three balanced random variables (Janas, MEC, & Janssen 2022):

$$1 - \chi_{ab}^2 - \chi_{ac}^2 - \chi_{bc}^2 + 2\chi_{ab}\chi_{ac}\chi_{bc} \geq 0$$

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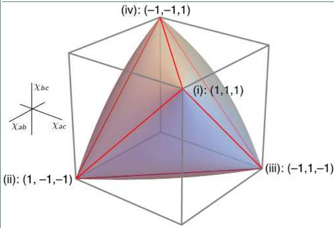

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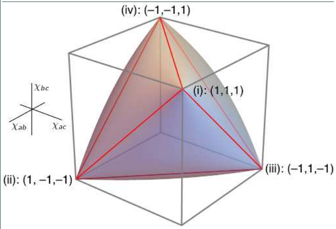
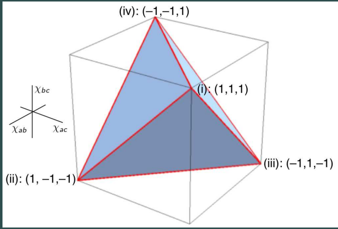


Classical tetrahedron:



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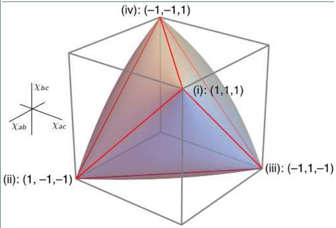

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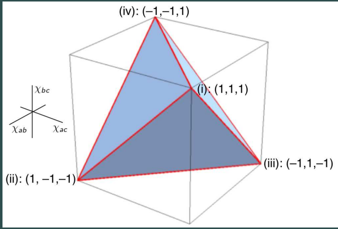

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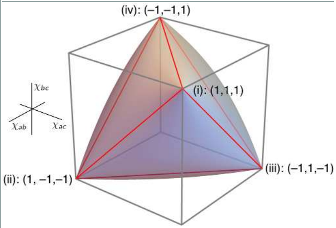
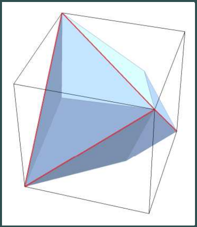


Classical polytope (spin 1):



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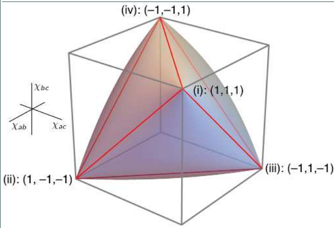
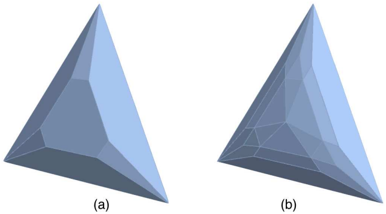


Classical polytope (spin $\frac{3}{2}$ and 2):



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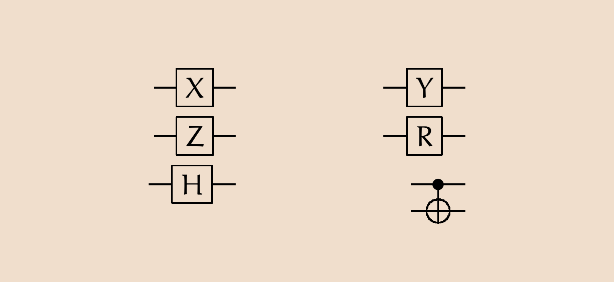
- The Clifford group of operations alone are sufficient to generate an entangled state.
- However: You also need to use it effectively

Clifford group of quantum logic gates:



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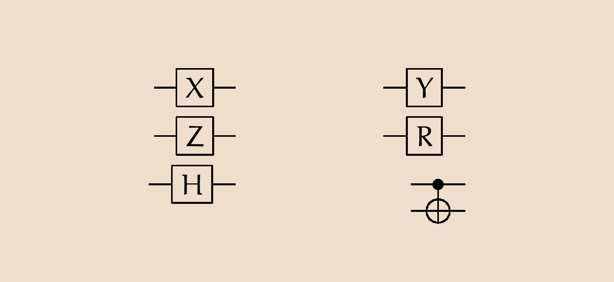
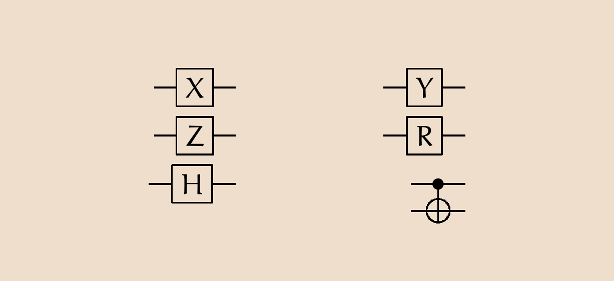
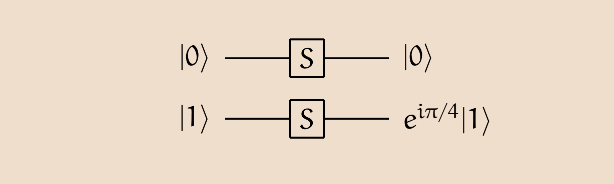


Not universal

Universal set of quantum logic gates:





Can simulate any quantum gate to arbitrary accuracy

Analogy: CHSH inequality (MEC 2017)

Assuming that outcomes of local experiments depend only on the local setup and on the value of a hidden variable λ assigned to the combined system at state preparation, then:

$$|\langle \sigma_m \otimes \sigma_n \rangle + \langle \sigma_m \otimes \sigma_{n'} \rangle| + |\langle \sigma_{m'} \otimes \sigma_n \rangle - \langle \sigma_{m'} \otimes \sigma_{n'} \rangle| \leq 2.$$

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But no conflict when $\hat{m}$ and $\hat{n}$, $\hat{m}$ and $\hat{n}'$, $\hat{m}'$ and $\hat{n}$, and $\hat{m}'$ and $\hat{n}'$ are all oriented at angles $\propto \pi/2$.

$\Rightarrow$ The Pauli measurements (Clifford group).

Kinematical constraints (broad sense; see Janssen 2009):

- Constraints imposed by a theoretical framework on our physical description of a system independently of the specifics of its dynamics.
- Kinematics of QM are less restrictive.

Independent of the specifics $\neq$ independent of the dynamics per se.

Gottesman-Knill Theorem:

- Merely possessing an entangled state isn't enough to yield a speedup over classical computation.
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- However: You also need to use it effectively
- I.e., you need to perform the right operations (duh!)

Thanks!

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